# Numerical analysis of the $\pi^+\pi^-$ atom lifetime in ChPT

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**Abstract**

We apply Chiral Perturbation Theory at one loop to analyze the general formula for the $\pi^+\pi^-$ atom lifetime derived recently in the framework of QCD [1]. The corresponding analytic expression is investigated numerically, and compared with recent work in the literature.

**1.** The DIRAC collaboration at CERN [2] aims to measure the lifetime of the $\pi^+\pi^-$ atom (pionium) in its ground state at the 10% level. This atom decays predominantly into two neutral pions, $\Gamma = \Gamma_{2\pi^0} + \Gamma_{2\gamma} + \ldots$, with $\Gamma_{2\gamma}/\Gamma_{2\pi^0} \sim 4 \cdot 10^{-3}$ [2]. The measurement of $\Gamma_{2\pi^0}$ allows one [1,3–5] to determine the difference $a_0 - a_2$ of the strong $S$-wave $\pi\pi$ scattering lengths with isospin $I = 0, 2$. One may then confront the predictions for this quantity obtained in standard ChPT [6,7] with the lifetime measurement, and furthermore analyze the nature of spontaneous chiral symmetry breaking in QCD [8]. In order to perform these investigations, one needs to know the theoretical expression for the width of pionium with a precision that properly matches the accuracy of

the lifetime measurement of DIRAC. It is the aim of the present article to provide the necessary numerical setting.

**2.** The formation of the atom and its subsequent decay into two neutral pions is induced by isospin breaking effects in the underlying theory. In the present framework (QCD including photons), these are the electromagnetic interactions and the mass difference of the up and down quarks. In the following, it is useful to count $\alpha \simeq 1/137$ and $(m_d - m_u)^2$ as small parameters of order $\delta$.

More than forty years ago, Deser *et al.* [3] derived the formula for the width of the $\pi^- p$ atom at leading order in isospin symmetry breaking. Later in Refs. [4,5], this result was adapted to the $\pi^+ \pi^-$ atom[1]. In particular, it was shown that - again at leading order in isospin symmetry breaking effects - the width $\Gamma_{2\pi^0}^{\text{LO}}$ of pionium is proportional to the square of the difference $a_0 - a_2$,

$$\Gamma_{2\pi^0}^{\text{LO}} = \frac{2}{9} \alpha^3 p^\star (a_0 - a_2)^2 \; ; \; p^\star = (M_{\pi^+}^2 - M_{\pi^0}^2 - \frac{1}{4} M_{\pi^+}^2 \alpha^2)^{1/2}. \tag{1}$$

At leading order in $\delta$, the momentum $p^\star$ becomes $\sqrt{2M_{\pi^+}(M_{\pi^+} - M_{\pi^0})}$ - this is the expression used in [4,5]. We prefer to use Eq. (1), because in this manner, one disentangles the kinematical corrections - due to the expansion of the square root - from true dynamical ones.

In our recent article [1], we derived a general expression for the pionium lifetime, that is valid at leading and next-to-leading order in isospin breaking,

$$\Gamma_{2\pi^0} = \frac{2}{9} \alpha^3 p^\star \mathcal{A}^2 (1 + K). \tag{2}$$

The quantities $\mathcal{A}$ and $K$ are expanded in powers of $\delta$. In particular, it has been shown in [1] that[2]

$$\mathcal{A} = -\frac{3}{32\pi} \text{Re} A_{\text{thr}}^{+-00} + o(\delta), \tag{3}$$

where $\text{Re} A_{\text{thr}}^{+-00}$ is calculated as follows. One evaluates the relativistic scattering amplitude for the process $\pi^+ \pi^- \to \pi^0 \pi^0$ at order $\delta$ near threshold and removes the (divergent) Coulomb phase. The real part of this matrix element develops singularities that behave like $|\mathbf{p}|^{-1}$ and $\ln 2|\mathbf{p}|/M_{\pi^+}$ near threshold ($\mathbf{p}$

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[1] There are a few misprints in Eq. (6) for the pionium decay rate in Ref. [4]. The correct result is displayed in Ref. [5].
[2] We use throughout the Landau symbols $O(x)$ $[o(x)]$ for quantities that vanish like $x$ [faster than $x$] when $x$ tends to zero. Furthermore, it is understood that this holds modulo logarithmic terms, i.e. we write also $O(x)$ for $x \ln x$.

denotes the center of mass momentum of the charged pions). The remainder, evaluated at the $\pi^+\pi^-$ threshold $\mathbf{p} = \mathbf{0}$, equals $\mathrm{Re}A_{\mathrm{thr}}^{+-00}$. It contains terms of order $\delta^0$ and $\delta$ and is normalized such that, in the isospin symmetry limit, $\mathcal{A} = a_0 - a_2$. Finally, the quantity $K$ starts at order $\alpha \ln \alpha$ - its explicit expression up to and including terms of order $\delta$ is given by

$$
K = \frac{\Delta_\pi}{9M_{\pi^+}^2} (a_0 + 2a_2)^2 - \frac{2\alpha}{3} (\ln \alpha - 1) (2a_0 + a_2) + o(\delta),
$$

$$
\Delta_\pi = M_{\pi^+}^2 - M_{\pi^0}^2. \tag{4}
$$

In the derivation of Eqs. (2) - (4), the chiral expansion has not been used - details will be provided in a forthcoming publication [9].

The following remarks are in order. First, it turns out that numerically, the correction factor $(1 + K)$ is very close to unity, see below. The present uncertainties in the values of the scattering lengths that enter the expression for $K$ are thus of no significance in this respect. Second, it follows that the measurement of the pionium lifetime amounts to a precise measurement of the quantity $\mathcal{A}$, that is not affected by use of any chiral expansion.

**3.** In order to extract $\pi\pi$ scattering lengths from experimental information on the width, we invoke ChPT and relate the amplitude $\mathcal{A}$ and the difference of the $S$-wave scattering lengths $a_0 - a_2$ order by order in the chiral expansion. This may be achieved as follows. First, we expand $\mathcal{A}$ in powers of the isospin breaking parameter $\delta$. Because the scattering amplitude $\pi^+\pi^- \to \pi^0\pi^0$ in QCD does not contain terms linear in the quark mass difference $m_u - m_d$, we write

$$
\mathcal{A} = a_0 - a_2 + h_1 (m_d - m_u)^2 + h_2 \alpha + o(\delta), \tag{5}
$$

where $a_0$ and $a_2$ denote, *by definition*, the strong $\pi\pi$ scattering lengths evaluated in QCD at $e = 0$, $m_u = m_d$. The quark masses are tuned such that the pion mass in the isospin-symmetric world coincides with the charged pion mass. The values of other parameters are not changed when performing this isospin symmetry limit. For illustration, the difference $a_0 - a_2$ at one loop is given by [7]

$$
a_0 - a_2 = \frac{9M_{\pi^+}^2}{32\pi F^2} \left(1 + \frac{M_{\pi^+}^2}{288\pi^2 F^2} \left\{33 + 8\bar{l}_1 + 16\bar{l}_2 - 3\bar{l}_3\right\}\right) + O(M_{\pi^+}^6), \tag{6}
$$

where $\bar{l}_i$ denotes the running coupling constant $l_i^r$ at scale $\mu = M_{\pi^+}$, and $F$ stands for the pion decay constant in the chiral limit, $F \sim 88 \ MeV$ [6].

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Second, the coefficients $h_i$ are evaluated in the framework of chiral perturbation theory. This calculation is straightforward - one only needs to determine in the $\pi^+\pi^- \to \pi^0\pi^0$ amplitude the terms proportional to $\alpha$ and $(m_d - m_u)^2$. Fortunately, the result at order $e^2p^2$ is already available in the literature [10]. At this order, $h_1$ vanishes. Indeed, $h_1$ is at least of order $\hat{m}$ in any order of the chiral expansion, with $\hat{m} = (m_u + m_d)/2$. The expression for $h_2$ can be read off from amplitude given in Ref. [10]. The result is

$$h_1 = O(\hat{m})\,,$$

$$h_2 = h_\Delta + h_\gamma + O(\hat{m}^2)\,,$$

$$h_\Delta = \frac{3\Delta_\pi^{e.m.}}{32\pi\alpha F^2}\left\{1 + \frac{M_{\pi^+}^2}{12\pi^2 F^2}\left[\frac{23}{8} + \bar{l}_1 + \frac{3}{4}\,\bar{l}_3\right]\right\}\,,$$

$$h_\gamma = \frac{3M_{\pi^+}^2}{256\pi^2 F^2}\,p(k_i)\,. \tag{7}$$

Here, $p(k_i)$ denotes a particular combination of electromagnetic low-energy constants $k_i^r$ that occur in the effective Lagrangian of $SU(2) \times SU(2)$ at order $e^2p^2$ [10,11]. Using the Lagrangian of Ref. [10], one has

$$p(k_i) = -30 + 9\bar{k}_1 + 6\bar{k}_3 + 2\bar{k}_6 + \bar{k}_8 + \frac{4}{3}Z(\bar{k}_1 + 2\bar{k}_2 + 6\bar{k}_4 + 12\bar{k}_6 - 6\bar{k}_8)\,, \tag{8}$$

and

$$\Delta_\pi^{e.m.} = \Delta_\pi\big|_{m_u=m_d}\,, \qquad Z = \frac{\Delta_\pi}{8\pi\alpha F^2}\bigg|_{m_u=m_d=0}\,. \tag{9}$$

The quantities $\bar{k}_i$ denote again the running couplings $k_i^r$ at scale $\mu = M_{\pi^+}$. Note that according to our counting, the quantity $L_\pi = \ln(M_{\pi^+}^2/M_{\pi^0}^2)$ introduced in Ref. [10], is of order $\delta$ and hence does not contribute to $h_2$.

**4.** In the numerical analysis, it is convenient to relate the particular combination of electromagnetic low-energy constants $\bar{k}_i$ that appears in Eq. (8), to the low-energy constants $K_i^r$ in the $SU(3) \times SU(3)$ version of ChPT. Estimates for the numerical values of $K_i^r$ are available in the literature [12–14], whereas the $\bar{k}_i$ have not yet been determined to the best of our knowledge. The relation between $\bar{k}_i$ and $K_i^r$ is straightforward to work out: one e.g. evaluates the amplitude $\pi^+\pi^- \to \pi^0\pi^0$ and $M_{\pi^0}^2$ in the framework of $SU(3) \times SU(3)$ and expands the result at small momenta and small quark masses, $p, m_u, m_d \ll m_s$. In this limit, the expression for the $SU(3) \times SU(3)$ amplitude goes over into the one given in [10], provided that one sets

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$$p(k_i) = P(K_i) - 8Z\bar{l}_4 \,,$$

$$P(K_i) = \frac{128\pi^2}{3}\left(-6(K_1^r + K_3^r) + 3K_4^r - 5K_5^r + K_6^r + 6(K_8^r + K_{10}^r + K_{11}^r)\right)$$

$$- (18 + 28Z)\ln\frac{M_{\pi^+}^2}{\mu^2} - 2Z\left(\ln\frac{m_s B_0}{\mu^2} + 1\right) - 30\,. \tag{10}$$

Here, $B_0$ is related to the chiral condensate in $SU(3) \times SU(3)$ and $m_s$ denotes the strange quark mass [6]. Furthermore, we have used that, at this order in the low-energy expansion, $Z$ equals its $SU(3) \times SU(3)$ analog $Z_0$. The couplings $K_i^r$ can be expressed [13] as a convolution of a QCD correlation function with the photon propagator, plus a contribution from QED counterterms. We have checked that $P(k_i)$ is independent of the QCD scale $\mu_0$ that must be introduced in the QCD Lagrangian after taking into account electromagnetic effects [13,14][3].

Using the relation between $F$ and $F_\pi$ in QCD [6],

$$F_\pi = F\left(1 + \frac{M_{\pi^+}^2}{16\pi^2 F^2}\bar{l}_4 + O(\hat{m}^2)\right), \tag{11}$$

the decay width of pionium can finally be expressed in the form

$$\Gamma_{2\pi^0} = \frac{2}{9}\alpha^3 p^* \left(a_0 - a_2 + \epsilon\right)^2 (1 + K)\,, \tag{12}$$

where

$$\epsilon = \alpha h_\Delta + \alpha h_\gamma + \cdots$$

$$= \frac{3\Delta_\pi^{e.m.}}{32\pi F_\pi^2}\left\{1 + \frac{M_{\pi^+}^2}{12\pi^2 F_\pi^2}\left[\frac{23}{8} + \bar{l}_1 + \frac{3}{4}\bar{l}_3 + \frac{3}{2}\bar{l}_4\right]\right\}$$

$$+ \frac{3\alpha M_{\pi^+}^2}{256\pi^2 F_\pi^2}\left(P(K_i) - 8Z\bar{l}_4\right) + \cdots\,. \tag{13}$$

The ellipses denote terms of $O(\hat{m}(m_d - m_u)^2, \alpha\hat{m}^2)$ and of $o(\delta)$. We expect these terms to give a negligible contribution to $\epsilon$. Note that the term proportional to $\bar{l}_4$ cancels in the sum $\alpha h_\Delta + \alpha h_\gamma$ at $O(e^2 p^2)$. The formulae (12) and (13) will be used below for the numerical analysis.

**5.** For the numerical analysis of Eqs. (12), (13), we use the following values for the various quantities that occur in these expressions. First, we recall

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[3] We thank B. Moussallam for clarifying remarks concerning this point.

that the non-electromagnetic part of the pion mass difference is tiny, of order $\sim 0.1$ MeV [15]. Therefore, we identify $\Delta_\pi^{e.m.}$ with the experimentally measured total shift $\Delta_\pi$. Similarly, the value of $Z$ is determined from Eq. (9) using the observed value of $\Delta_\pi$. Further, in the calculations we replace $m_s B_0$ by $M_{K^+}^2 - M_{\pi^+}^2/2$, according to our definition of the isospin symmetry limit. The values used for the low-energy constants are: $F_\pi = 93.2$ MeV [6,16], $\bar{l}_1 = -2.3 \pm 3.7$, $\bar{l}_2 = 6.0 \pm 1.3$, $\bar{l}_3 = 2.9 \pm 2.4$ [6], $\bar{l}_4 = 4.4 \pm 0.3$ [17]. For $K_i^r(\mu)$, we use the values given by Baur and Urech in Ref. [12, Table 1]: $K_1^r = -6.4$, $K_3^r = 6.4$, $K_4^r = -6.2$, $K_5^r = 19.9$, $K_6^r = 8.6$, $K_8^r = K_{10}^r = 0$, $K_{11}^r = 0.6$ (in units of $10^{-3}$). We evaluate $P(K_i)$ at scale $\mu = M_\rho$. Further, we attribute an uncertainty $2/16\pi^2$ - that stems from dimensional arguments - to each $K_i^r$. The values of $K_i^r$ obtained both by Moussallam [13] and by Bijnens and Prades [14], lie then within the uncertainties attributed. In the final expression for $\epsilon$, the uncertainties coming from $\bar{l}_i$ and $K_i^r$, are added quadratically. Finally, we use for $a_0$ and $a_2$ the values corresponding to set 2 in Ref. [16]: $a_0 = 0.206$, $a_2 = -0.0443$, without attributing any error. These scattering lengths enter only the correction $K$ in (12), which is very small.

Inserting all this in $\epsilon$, and omitting the terms of higher order in the chiral expansion and in isospin violating effects [indicated by the ellipses in (13)], we arrive at

$$\epsilon = (0.58 \pm 0.16) \cdot 10^{-2}, \qquad K = 1.07 \cdot 10^{-2}. \tag{14}$$

The error in this result includes *only* the uncertainty in the values of the low-energy constants $K_i^r$ and $\bar{l}_i$. Resonance saturation introduces a scale dependence in the final result for $\epsilon$. If saturation is assumed at $\mu = 500$ MeV ($\mu = 1$ GeV), we find $\epsilon = 0.51 \cdot 10^{-2}$ ($\epsilon = 0.62 \cdot 10^{-2}$). Eqs. (12) - (14) are the main result of this letter. We add the following remarks concerning this analysis.

i) The term $\alpha h_\Delta$ ($\alpha h_\gamma$) contributes with $0.51 \cdot 10^{-2}$ ($0.07 \cdot 10^{-2}$) to $\epsilon$, see Eq. (13). Therefore, $\alpha h_\Delta$ is by far the dominant effect in the improvement of the leading order formula (1). The dominant contribution in $K$ stems from the logarithmic term $\sim \alpha \ln \alpha$.

ii) At one-loop order, $\alpha h_\Delta$ can be written in the following form,

$$\alpha h_\Delta = \frac{\Delta_\pi^{e.m.}}{3M_{\pi^+}^2} \left\{ a_0 - a_2 + \frac{M_{\pi^+}^4}{256\pi^3 F_\pi^4} \left[ 9 + 4(\bar{l}_1 - \bar{l}_2) + \frac{21}{4}\bar{l}_3 \right] \right\}. \tag{15}$$

For the values of the low-energy constants $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_3$ given above, the second term in the curly bracket turns out to be very small, of order $-6 \cdot 10^{-3}$. Consequently, the one-loop correction to the ratio $\alpha h_\Delta/(a_0 - a_2)$ is tiny, and the latter is dominated by the tree-level contribution $\Delta_\pi/3M_{\pi^+}^2$.

iii) If one is willing to rely on the numerical values of $a_0$ and $a_2$ displayed above, one may evaluate the lifetime of pionium in the ground state: $\tau = 3.25 \cdot 10^{-15}$ s. The correction to the lowest-order formula (1) by Deser *et al.* then becomes $(\Gamma_{2\pi^0} - \Gamma_{2\pi^0}^{\mathrm{LO}})/\Gamma_{2\pi^0}^{\mathrm{LO}} = 0.058$. We prefer not to attach any error to these numbers, because they are based on the numerical values of $a_0$ and $a_2$, and these are not yet known with sufficient accuracy.

**6.** We now compare our result with recent work in the literature and start the discussion with Refs. [18], where hadronic atoms have been studied in the framework of potential scattering theory. The correction to the leading-order formula for the width has been worked out in [18] numerically - a comparison with our analytic result is thus not possible. The numerical result quoted in [18] differs significantly from ours: isospin violating corrections to the leading order result increase the lifetime according to these authors, in contrast to Eqs. (12), (14). The discrepancy does not come as a surprise - in the present form, the potential approach [18] does not reproduce all isospin-breaking terms that are present in the Standard Model. The leading correction in ChPT stems from tuning the quark mass in the isospin-symmetric phase such that the pion mass in the isospin-symmetric world coincides with the charged pion mass. Since the pion-pion interaction depends on the quark mass, this effectively leads to a change in the $\pi\pi$ potential when the isospin limit is considered. In addition, the potential model in [18] does not take into account the direct quark-photon effects encoded in the low-energy constants $\bar{k}_i$ in our approach. We believe that, for a consistent calculation of the corrections to the $\pi^+\pi^-$ atom decay width, the potential in the scattering theory approach should be matched to ChPT in the *isospin violating* phase. This procedure would guarantee that the above mentioned effects are included.

**7.** In order to present a coherent comparison with other calculations performed in the framework of Quantum Field Theory, it is useful to expand also the quantity $K$, similarly to $\mathcal{A}$ in (5),

$$K = f_1 \, (m_d - m_u)^2 + f_2 \, \alpha \ln \alpha + f_3 \, \alpha + o(\delta), \tag{16}$$

and to rewrite the decay width (2) in the form

$$\Gamma_{2\pi^0} = \frac{2}{9} \, \alpha^3 \, p^\star R^2 \,,$$

$$R = a_0 - a_2 + r_1 \alpha \ln \alpha + r_2 \alpha + r_3 (m_d - m_u)^2 + o(\delta) \,, \tag{17}$$

where the coefficients in the expansion are given by

$$r_1 = \frac{1}{2} \, (a_0 - a_2) f_2 \,,$$

$$r_2 = h_2 + \frac{1}{2}(a_0 - a_2)f_3 \,,$$

$$r_3 = h_1 + \frac{1}{2}(a_0 - a_2)f_1 \,. \tag{18}$$

The comparison with other approaches becomes then rather easy: one compares the analytic expressions for $r_i$ at a given order in the chiral expansion.

The correction to the pionium decay width were evaluated in Refs. [19] by use of 3D constraint theory equations, and in Refs. [20] in the Bethe-Salpeter approach. The term proportional to $\alpha^2/\Delta_\pi$ that emerges from the expansion of $p^\star$, was omitted in the final expressions for the decay width in these papers. Note that though this term is algebraically of order $\delta$, the numerical effect coming from it is negligible. Further, rewriting the expressions found in these works in the form (17,18), we find that the coefficient $r_1$ coincides to all orders in the chiral expansion with our result, whereas $r_2$ coincides with the above result up to and including terms of order $p^2$. The matching relation (10) does however not agree with the one proposed in Refs. [19]. The expressions in Refs. [19] and [20] contain some of the higher-order terms in the quark mass expansion and in isospin breaking effects. These cannot be reliably predicted based on $O(e^2p^2)$ calculations alone. The effect of those terms is however small. The numerical values of the corrections in [19,20] almost coincide with our result.

Labelle and Buckley [21] were the first to apply the non relativistic effective Lagrangian approach to the problem of hadronic atoms. They evaluated the leading order term, and in addition the correction due to electron vacuum polarization. This contribution is analytically of order $\alpha^2$ and thus beyond the accuracy considered here. Although this correction is potentially large, because it is proportional to $(M_{\pi^+}/m_{electron})\alpha^2 \simeq 2\alpha$, it amounts to a tiny contribution $3.9 \cdot 10^{-4}$ to $\epsilon$, which one may safely ignore. The vacuum polarization correction to the $\pi^+\pi^-$ atom decay width was obtained independently in Ref. [22] - the result agrees to that of [21]. Note that the part of the correction due to vacuum polarization, with no Coulomb corrections in the intermediate state, was considered also in Refs. [20], and the result agrees with the corresponding one given in Ref. [21].

Recently, Kong and Ravndal [23,24] and Holstein [26] have applied non relativistic effective Lagrangian techniques to this problem as well. In Refs. [23,24], Coulomb corrections are not taken into account, which amounts to $r_1 = 0$. The relativistic correction found in Ref. [24] amounts to rewriting the phase space factor $\sqrt{2M_{\pi^+}(M_{\pi^+} - M_{\pi^0})}$ as $\sqrt{M_{\pi^+}^2 - M_{\pi^0}^2}$ - this term is thus included in our approach. On the other hand, we do not agree in the value of the coefficient $r_2$ given in Ref. [23] already at leading order in the quark mass expansion. The discrepancy is due to the fact that - in [23] - the matching of the ef-

fective couplings in the non relativistic Lagrangian has not been carried out with a precision that is required to pin down all terms at this order in the expression for the width. If a complete matching at $O(\delta)$ is performed, agreement is achieved at the leading order in quark mass expansion [25]. The result of Ref. [26] for $r_2$ agrees with Ref. [23] - hence, it also shares its shortcomings. The contribution corresponding to $r_1$ is omitted in the final expression for the width in [26, Eq.(95)], although at previous stages in that work, the Coulomb corrections had been discussed. Finally, effects from $m_u \neq m_d$ are not disentangled in [23,24,26].

In Ref. [27], an effective non relativistic Lagrangian was used to derive the expression for the decay width in terms of effective couplings in that Lagrangian. The result agrees with ours [1, Eq.(11)]. The matching to the relativistic amplitude is carried out in [27] order by order in the chiral expansion. We agree with the result of [27] for the coefficient $r_1$ at leading order in the chiral expansion. The result for the coefficient $r_2$ is not explicitly given in [27], and a comparison is therefore not possible.

The result presented in Ref. [28] differs from ours in many respect. In particular, the expression for the decay width given in Ref. [28] contains an ultraviolet divergence that is regularized by introducing an explicit cutoff. A systematic renormalization procedure is not discussed. For this reason, the result of [28] can not be compared with the present one.

Finally, we comment on the work of Ref. [29]. First we note that in our approach, the decay width is calculated by considering the elastic $\pi^0\pi^0 \to \pi^0\pi^0$ amplitude as a function of the energy $E$ of the $\pi^0\pi^0$ pair. This amplitude develops a pole at $E = E_p$ on the second Riemann sheet - the imaginary part $\mathrm{Im}(E_p)$ is related to the width in the standard manner. As far as we can see, the approaches proposed in [23,24,26] amount to the same definition of the width, because it is identified in these works with the imaginary part of the level shift in Rayleigh-Schrödinger perturbation theory. On the other hand, Ref. [29] amounts to an alternative definition of $\Gamma_{2\pi^0}$ - a direct comparison with the present work is therefore not possible.

**8.** In conclusion, we have analyzed analytically and numerically the general formula [1] for the decay width of the pionium ground state at one-loop order in ChPT, and we have compared our result with other work available in the literature. We have in particular identified the reason for discrepancies of our result with other approaches. As we have shown, a precise determination of the pionium lifetime allows one to measure the amplitude $\mathcal{A}$ in Eq. (2), where no chiral expansion is used. By use of ChPT, one may determine the combination $a_0 - a_2$ of $S$-wave scattering lengths from Eqs. (12) - (14), that constitute the main result of this letter. As we expect the corrections due to higher orders in the chiral expansion to be small, we infer from this result that an accurate determination of $a_0 - a_2$ from a precise lifetime measurement is indeed feasible.

## References

[1] A. Gall, J. Gasser, V. E. Lyubovitskij, and A. Rusetsky, Phys. Lett. B 462 (1999) 335.

[2] B. Adeva *et al.*, CERN proposal CERN/SPSLC 95-1 (1995).

[3] S. Deser, M. L. Goldberger, K. Baumann, and W. Thirring, Phys. Rev. 96 (1954) 774.

[4] J. L. Uretsky and T. R. Palfrey, Jr., Phys. Rev. 121 (1961) 1798.

[5] S. M. Bilenky, Van Kheu Nguyen, L. L. Nemenov, and F. G. Tkebuchava, Sov. J. Nucl. Phys. 10 (1969) 469.

[6] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) 158 (1984) 142; Nucl. Phys. B 250 (1985) 465.

[7] J. Gasser and H. Leutwyler, Phys. Lett. B 125 (1983) 325; J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Phys. Lett. B 374 (1996) 210.

[8] M. Knecht, B. Moussallam, J. Stern, and N. H. Fuchs, Nucl. Phys. B 457 (1995) 513; ibid. B 471 (1996) 445.

[9] A. Gall, J. Gasser, V. E. Lyubovitskij, and A. Rusetsky, in preparation.

[10] M. Knecht and R. Urech, Nucl. Phys. B 519 (1998) 329.

[11] U.-G. Meißner, G. Müller, and S. Steininger, Phys. Lett. B 406 (1997) 154; ibid. B 407 (1997) 454 (E).

[12] R. Baur and R. Urech, Nucl. Phys. B 499 (1997) 319.

[13] B. Moussallam, Nucl. Phys. B 504 (1997) 381.

[14] J. Bijnens and J. Prades, Nucl. Phys. B 490 (1997) 239.

[15] J. Gasser and H. Leutwyler, Phys. Rep. 87 (1982) 77.

[16] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Nucl. Phys. B 508 (1997) 263.

[17] J. Bijnens, G. Colangelo, and P. Talavera, JHEP 9805 (1998) 014.

[18] U. Moor, G. Rasche, and W. S. Woolcock, Nucl. Phys. A 587 (1995) 747; A. Gashi, G. C. Oades, G. Rasche, and W. S. Woolcock, Nucl. Phys. A 628 (1998) 101; G. Rasche and A. Gashi, preprint hep-ph/9807564.

[19] H. Jallouli and H. Sazdjian, Phys. Rev. D 58 (1998) 014011; H. Sazdjian, preprint hep-ph/9809425.

[20] V. E. Lyubovitskij and A. G. Rusetsky, Phys. Lett. B 389 (1996) 181; V. E. Lyubovitskij, E. Z. Lipartia, and A. G. Rusetsky, JETP Lett. 66 (1997) 783; M. A. Ivanov, V. E. Lyubovitskij, E. Z. Lipartia, and A. G. Rusetsky, Phys. Rev. D 58 (1998) 094024.

[21] P. Labelle and K. Buckley, preprint hep-ph/9804201.

[22] U. Jentschura, G. Soff, V. Ivanov, and S. G. Karshenboim, Phys. Lett. A 241 (1998) 351.

[23] X. Kong and F. Ravndal, Phys. Rev. D 59 (1999) 014031.

[24] X. Kong and F. Ravndal, preprint hep-ph/9905539.

[25] F. Ravndal (private communication), and revised version of Ref. [24].

[26] B. R. Holstein, preprint nucl-th/9901041.

[27] D. Eiras and J. Soto, preprint hep-ph/9905543.

[28] G. G. Bunatian, Nucl. Phys. A 645 (1999) 314.

[29] P. Minkowski, preprint hep-ph/9808387.